Critical Points of Toroidal Belyĭ Maps

Tesfa Asmara (Pomona College), Erik Imathiu-Jones (California Institute of Technology), Maria Maalouf (California State University at Long Beach), Isaac Robinson (Harvard University), and Sharon Sneha Spaulding (University of Connecticut)

Pomona Research in Mathematics Experience (PRiME)

Abstract

A Belyĭ map $\beta : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ is a rational function with at most three critical values; we may assume these values are $\{0, 1, \infty\}$. Replacing $\mathbb{P}^1$ with an elliptic curve $E: \; y^2 = x^3 + A\,x + B$, there is a similar definition of a Belyĭ map $\beta : E(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$. Since $E(\mathbb{C}) \simeq \mathbb{T}^2(\mathbb{R})$ is a torus, we call (E, β) a Toroidal Belyĭ pair.

There are many examples of Belyĭ maps $\beta : E(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ associated to elliptic curves; several can be found online at LMFDB. Given such a Toroidal Belyĭ map of degree N, the inverse image $G = \beta^{-1}(\{0, 1, \infty\})$ is a set of N elements which contains the critical points of the Belyĭ map. In this project, we investigate when G is contained in $E(\mathbb{C})_{\text{tors}}$.

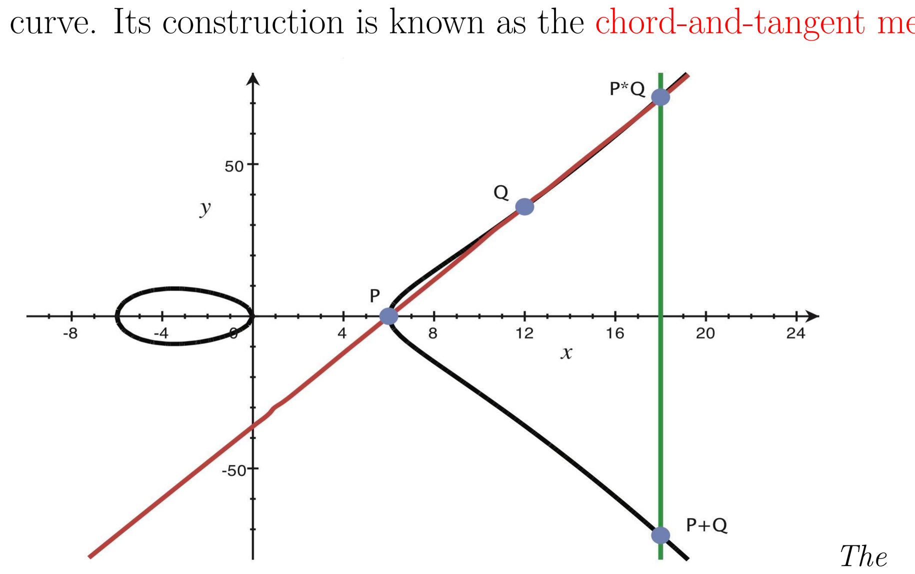
This is work done as part of the Pomona Research in Mathematics Experience (NSA H98230-21-1-0015).

Elliptic Curves

- An elliptic curve, E, is a non-singular curve of genus one. In other words, it is a curve generated by an equation $f(x, y) = 0$ where
$$f(x, y) = y^2 + a_1\,x\,y + a_3\,y - (x^3 + a_2\,x^2 + a_4\,x + a_6)$$
and where all $a_i \in \mathbb{C}$ with O_E being the "point at infinity."
- The set of complex points on an elliptic curve $E(\mathbb{C})$ is a torus.

The Group Law on an Elliptic Curve

- There exists a binary operation $\oplus$ such that $(E(\mathbb{C}), \oplus)$ forms a group with O_E as the identity. This operation is known as the group law on the elliptic curve. Its construction is known as the chord-and-tangent method.



Chord-and-Tangent Construction

- An isogeny is a map $\psi : E \to X$ where E and X elliptic curves such that $\psi(P \oplus Q) = \psi(P) \oplus \psi(Q)$ for $P, Q \in E(\mathbb{C})$.

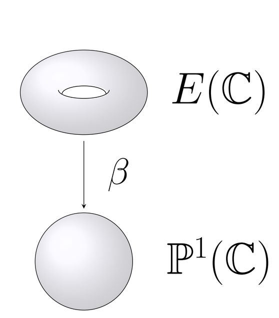


Figure 1: A Toroidal Belyĭ Map

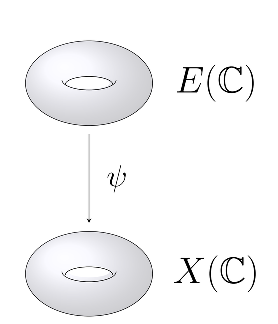


Figure 2: An isogeny

Critical Points and Toroidal Belyĭ Maps

Fix a rational function $\beta : E(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ where $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$.

- $P \in E(\mathbb{C})$ is a critical point if $\dfrac{\partial f}{\partial x}(P)\dfrac{\partial \beta}{\partial y}(P) - \dfrac{\partial f}{\partial y}(P)\dfrac{\partial \beta}{\partial x}(P) = 0$.
- $q \in \mathbb{P}^1(\mathbb{C})$ is a critical value if $q = \beta(P)$ for some critical point P.
- $Q \in E(\mathbb{C})$ is a quasi-critical point if $\beta(Q) = \beta(P)$ for critical point P.
- A Belyĭ map is function β as above with ≤ 3 critical values, $\{0, 1, \infty\}$.
- A Toroidal Belyĭ pair is a pair (E, β), where E is an elliptic curve and β is a Belyĭ map associated to E.

Examples of Toroidal Belyĭ Pairs (X, ϕ) with Quasi-Critical Points $\phi^{-1}(\{0, 1, \infty\}) \subseteq X(\mathbb{C})_{\text{tors}}$

LMFDB Label	Elliptic Curve X	Belyĭ Map $\phi : X(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$	Group Generated by $\phi^{-1}(\{0, 1, \infty\})$
3T1-3_3_3-a	$y^2 = x^3 + 1$	$\dfrac{1-y}{2}$	Z_3
4T1-4_4_2.2-a	$y^2 = x^3 - x$	x^2	$Z_2 \times Z_2$
4T5-4_4_3.1-a	$y^2 = x^3 + x^2 + 16\,x + 180$	$\dfrac{4\,y + x^2 + 56}{108}$	Z_8
5T4-5_5_3.1.1-a	$y^2 + x\,y = x^3 - 28\,x + 272$	$\dfrac{(x+13)\,y + 3\,x^2 + 4\,x + 220}{432}$	$Z_2 \times Z_{10}$
6T1-6_2.2.2_3.3-a	$y^2 = x^3 + 1$	$-x^3$	$Z_2 \times Z_6$
6T4-3.3_3.3_3.3-a	$y^2 = x^3 - 15\,x + 22$	$\dfrac{8\,(x-2)^2 - (x^2 - 4\,x + 7)\,y}{16\,(x-2)^2}$	Z_6
6T5-6_6_3.1.1.1-a	$y^2 = x^3 + 1$	$\dfrac{(1-y)\,(3+y)}{4}$	$Z_2 \times Z_6$
6T6-6_6_2.2.1.1-a	$y^2 = x^3 + 6\,x - 7$	$\dfrac{(x-1)^3}{27}$	$Z_2 \times Z_4$
6T7-4.2_4.2_3.3-a	$y^2 = x^3 - 10731\,x + 408170$	$\dfrac{11907\,(x-49)}{(x-7)^3}$	$Z_2 \times Z_4$
6T12-5.1_5.1_3.3-b	$y^2 + x\,y + y = x^3 + x^2 - 10\,x - 10$	$27\,\dfrac{(x+4)\,(2\,x^2 - 2\,x - 13) - (x+1)^2\,y}{(x^2 - x - 11)^3}$	$Z_2 \times Z_8$
6T12-5.1_5.1_5.1-a	$y^2 = x^3 + x^2 + 4\,x + 4$	$-16\,\dfrac{(x^2 - 2\,x - 4)\,y + 8\,(x+1)}{(x-4)\,x^5}$	Z_6
8T2-4.4_4.4_2.2.2.2-a	$y^2 = x^3 + x$	$\dfrac{(x+1)^4}{8\,x\,(x^2+1)}$	$Z_2 \times Z_4$
8T7-8_8_2.2.1.1.1.1-a	$y^2 = x^3 - x$	x^4	$Z_2 \times Z_4$

Example #1: 4T1-4_4_2.2-a

Consider the Toroidal Belyĭ pair (E, β) in terms of
$$E : y^2 = x^3 - x \qquad \text{and} \qquad \beta(x, y) = x^2.$$
The quasi-critical points are torsion:

Point	$(0,0)$	$(1,0)$	$(-1,0)$	O_E
Order	2	2	2	1

These points form a group:
$$\beta^{-1}(\{0, 1, \infty\}) = \{(0,0), (1,0), (-1,0), O_E\} \simeq Z_2 \times Z_2.$$

Example #2: 4T5-4_4_3.1-a

Consider the Toroidal Belyĭ pair (E, β) in terms of
$$E : y^2 = x^3 + x^2 + 16x + 180 \qquad \text{and} \qquad \beta(x, y) = (4y + x^2 + 56)/108.$$
The quasi-critical points are torsion:

Point	$(4,-18)$	$(22,-108)$	$(-2,12)$	O_E
Order	4	8	8	1

These points do not form a group.

Example #3: 5T5-5_4.1_4.1-a

Consider the Toroidal Belyĭ pair (E, β) in terms of
$$E : y^2 = x^3 + 5x + 10 \qquad \text{and} \qquad \beta(x, y) = ((x-5)y + 16)/32.$$
The quasi-critical points are not torsion:

Point	$(6,-16)$	$(1,4)$	$(6,16)$	$(1,-4)$	O_E
Order	∞	∞	∞	∞	1

These points do not form a group.

Motivating Questions

Given the following:

- (E, β), a Toroidal Belyĭ pair.
- $\Gamma = \beta^{-1}(\{0, 1, \infty\})$ as the set of quasi-critical points.

We ask the questions:

1. *When does Γ form a subgroup of $(E(\mathbb{C}), \oplus)$?*
2. The elements in Γ must be points with finite order whenever Γ is a group. *When are the points in Γ torsion elements in $E(\mathbb{C})$, regardless of Γ being a group?*

Theorem (PRiME 2021)

Given the following:

- (X, ϕ) a Toroidal Belyĭ pair, and $G = \phi^{-1}(\{0, 1, \infty\})$ as the set of quasi-critical points.
- $\beta = \phi \circ \psi$, where $\psi : E \to X$ is any non-constant isogeny, and $\Gamma = \beta^{-1}(\{0, 1, \infty\})$.

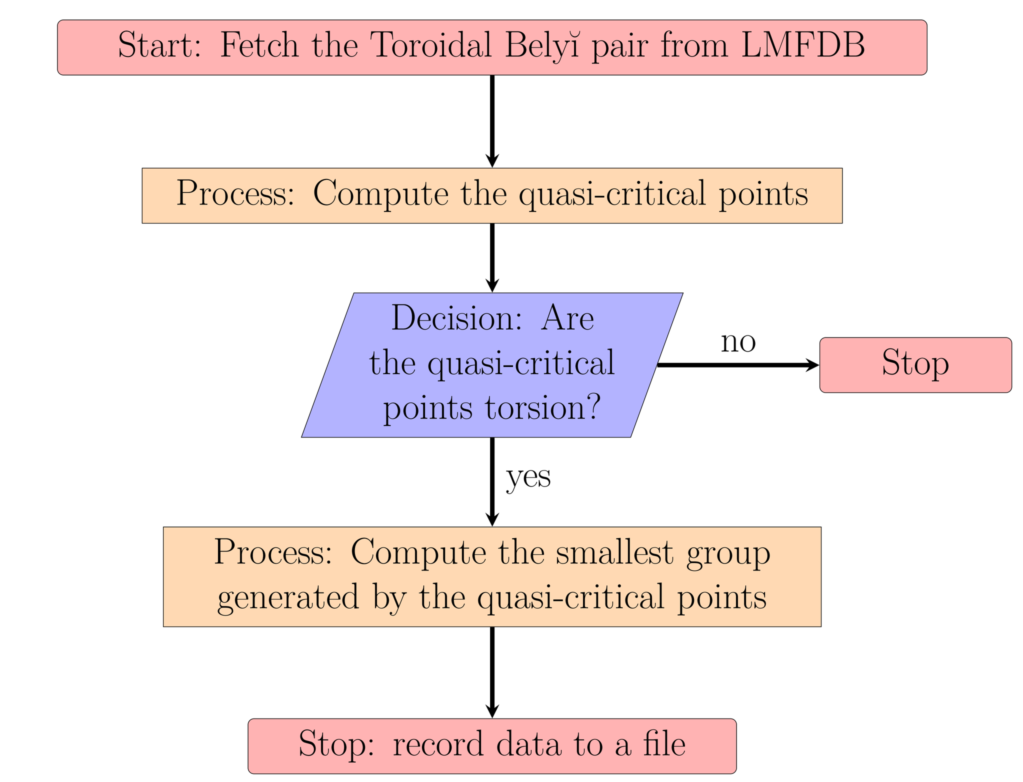
We have the main results:

1. (E, β) is a Toroidal Belyĭ pair.
2. Γ is contained in the torsion in $E(\mathbb{C})$ whenever G is contained in the torsion in $X(\mathbb{C})$.
3. Γ is a group whenever G is group.

Corollary

There are infinitely many Toroidal Belyĭ pairs where the set of quasi-critical points forms a group.

Computing Examples



Results from Computation

Degree of Belyĭ Map	Total from LMDFB	Total Number of Successfully Processed	Number with Quasi-Critical Points All Torsion
3	1	1 (100%)	1 (100%)
4	2	2 (100%)	2 (100%)
5	7	7 (100%)	1 (14%)
6	35	29 (83%)	7 (24%)
7	73	15 (21%)	0 (0%)
8	94	30 (32%)	2 (7%)
9	39	23 (59%)	0 (0%)
Totals	251	107 (43%)	13 (12%)

Future Work

- Modify the Sage code to run faster in order to get more examples.
- Find more examples of imprimitive Toroidal Belyĭ maps with quasi-critical points that form a group.
- Create an accessible website containing all the information on the data found.

References

[1] PRiME 2021 GitHub Repository. `https://github.com/PRiME-2021/Algorithms`

[2] Sage Mathematics Software System. `http://www.sagemath.org`

[3] Ernesto Girondo and Gabino González-Diez. *Introduction to Compact Riemann Surfaces and Dessins d'Enfants.* London Mathematical Society Student Texts, Volume 79 (2012).

[4] The L-functions and Modular Forms Database (LMFDB). `http://www.lmfdb.org`

[5] Joseph H. Silverman. *The Arithmetic of Elliptic Curves.* Graduate Texts in Mathematics, Volume 106. Second Edition (2009).

[6] Joseph H. Silverman and John T. Tate. *Rational points on elliptic curves.* Undergraduate Texts in Mathematics (1993).

Acknowledgements

- Dr. Edray Herber Goins (Pomona College)
- Dr. Alex Barrios (Carleton College)
- Dr. Rachel Davis (University of Wisconsin at Madison)
- Department of Mathematics, Pomona College
- National Security Agency (H98230-21-1-0015)